ON THE APPLICATION OF INTEGRAL AND VARIATIONAL PRINCIPLES OF MECHANICS TO PROBLEMS OF VIBRATIONS

(O PRIMENENII INTEGRAL'NYKH I VARIATSIONNYKH PRINTSIPOV MEKHANIKI V ZADACHAKH KOLEBANII)

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The variational principles of mechanics were formulated during the period of their establishment as requirements for certain functionals, defined by integrals, to attain a minimum or a maximum.

The question of the character of the extremum of these integrals or even regarding its existence is non-essential for the deduction of equations of motion of mechanical systems, which is carried out by means of equating to zero the first variation of the functional. It is therefore not surprising that together with real variational principles of mechanics, such as Hamilton's principle for conservative systems with holonomic constraints, there were established also integral principles formulated with the aid of integrals containing expressions with variations, but which did not, because of their structure, lead to a problem of the calculus of variations; an example of this type is the principle of Hamilton-Ostrogradskii for non-conservative systems.

As a result, the question of the character of the extremum in real variational principles became secondary and the establishment of the fact that in the principles of Hamilton-Ostrogradskii and Maupertuis one deals with a minimum for only sufficiently small intervals of time, did not have any influence on the basic practical applications of variational principles.

At the present time the corresponding terms belonging to the theory of kinetic foci [1] are almost never mentioned either in the educational or in the scientific literature of mechanics.

One can however indicate problems in which the use of minimum properties of action according to Hamilton, without regard to results of the theory of kinetic foci, leads to erroneous interpretations.

As an example one can mention certain problems of the theory of vibrations which is reducible to the determination of eigenvalues and eigenfunctions.

In Book [2] and in Paper [3] there is a natural attempt to establish extremum properties for frequencies and mode shapes in problems of vibrations of mechanical systems on the basis of extremum properties of action as defined by Hamilton.

The neglect of the theory of kinetic foci led Biezeno and Grammel [2] to the necessity of distorting the formulation of the principle (introduction of $L = \Pi - T$ instead of $L = -\Pi$), while Poschl [3] was led to contradictions in his discussion (replacement of a minimum by a maximum) which obtained after he himself had corrected the indicated error [4].

We investigate below the question of the possibility of using variational and integral principles to determine frequencies and mode shapes in vibrating elastic systems and show that the principle of Hamilton-Ostrogradskii, in changed formulation, makes it possible to reduce the problem to that of investigating the stationary values of a certain functional without, however, allowing us to draw any conclusion regarding the character of its extremum.

This latter information should be obtained on the basis of a separate study which is not connected with variational and integral principles of mechanics.

1. Let us consider a material system subjected to conservative forces (it is assumed that the constraints are stationary and holonomic). Let q_s and p_s designate the generalized coordinates and momenta respectively; H is Hamilton's function expressed through these variables. In passing to an infinitely close motion in which the coordinates and momenta will be $q_s + x_s$ and $p_s + u_s$, its increment ΔH is equal to

$$\Delta H = \delta H + \delta^2 H + \ldots = \sum_{s=1}^{n} \left(\frac{\partial H}{\partial q_s} x_s + \frac{\partial H}{\partial p_s} u_s\right) + \\ + \Omega(x_1, \ldots, x_n, u_1, \ldots, u_n) + \ldots \qquad (1.1)$$

Here Ω indicates the homogeneous quadratic form

$$\Omega = \frac{1}{2} \sum_{s=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial^2 H}{\partial q_s \partial q_k} x_s x_k + 2 \frac{\partial^2 H}{\partial q_s \partial p_k} x_s u_k + \frac{\partial^2 H}{\partial p_s \partial p_k} u_s u_k\right) \qquad (1,2)$$

The quadratic form entering into this system

$$\delta^2 T' = \frac{1}{2} \sum_{s=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 H}{\partial p_s \partial p_k} u_s u_k \qquad (1.3)$$

is conjugate to the quadratic form

$$\delta^2 T = \frac{1}{2} \sum_{s=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 T}{\partial \dot{q}_s \partial \dot{q}_k} \dot{x}_s \dot{x}_k \tag{1.4}$$

where T is the kinetic energy of the system. Therefore, the form (1.3) is positive definite. The system of linear equations in variations for the canonical system of equations of motion

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \qquad \dot{p}_s = -\frac{\partial H}{\partial q_s} \qquad (s = 1, \ldots, n) \tag{1.5}$$

may also be written in canonical form:

$$\begin{aligned} \dot{x}_s &= \sum_{k=1}^{n} \left(\frac{\partial^2 H}{\partial q_k \partial p_s} x_k + \frac{\partial^2 H}{\partial p_k \partial p_s} u_k \right) = \frac{\partial \Omega}{\partial u_s} \\ \dot{u}_s &= -\sum_{k=1}^{n} \left(\frac{\partial^2 H}{\partial q_k \partial q_s} x_k + \frac{\partial^2 H}{\partial p_k \partial q_s} u_k \right) = -\frac{\partial \Omega}{\partial x_s} \end{aligned} \qquad (s = 1, \ldots, n) \tag{1.6}$$

If it is assumed that Cauchy's integral is known for the system (1.5)

$$q_s = q_s(t - t_0, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n), \quad p_s = p_s(t - t_0, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) \tag{1.7}$$

where α_k and β_k are the initial values of the coordinates and the momenta, then the functions of time

$$\xi_s^{(m)} = \frac{\partial q_s}{\partial \alpha_m}, \qquad \eta_s^{(m)} = \frac{\partial q_s}{\partial \beta_m}, \qquad \zeta_s^{(m)} = \frac{\partial p_s}{\partial \alpha_m}, \qquad \vartheta_s^{(m)} = \frac{\partial p_s}{\partial \beta_m} \tag{1.8}$$
$$(s, m = 1, 2, \ldots, n)$$

give, as is known, a system of particular solutions of the variational equations (1.6). Their linear dependence is a consequence of the equality

$$D\left(\frac{q_1, \ldots, q_n, p_1, \ldots, p_n}{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n}\right) = 1 \tag{1.9}$$

which expresses the invariance of an element of volume of phase space (Liouville's theorem).

In as much as relations (1.7) represent Cauchy's integral of the system (1.5), the following relations are valid

$$\xi_s^{(m)}(t_0) = \vartheta_s^{(m)}(t_0) = \delta_{sm}, \qquad \eta_s^{(m)}(t_0) = \zeta_s^{(m)}(t_0) = 0 \tag{1.10}$$

where δ_{sm} are Kronecker's symbols. Therefore, Cauchy's integral of the system of equations in variations (1.6) will be

$$\begin{aligned} x_s(t) &= \sum_{m=1}^{n} [x_m(t_0)\, \xi_s^{(m)}(t) + u_m(t_0)\, \eta_s^{(m)}(t)] \\ u_s(t) &= \sum_{m=1}^{n} [x_m(t_0)\, \zeta_s^{(m)}(t) + u_m(t_0)\, \vartheta_s^{(m)}(t)] \end{aligned} \tag{1.11}$$

2. The motion of the system defined by the Equations (1.7) shall be called its direct path C_0; let $q_s + \delta p_s$ and $p_s + \delta p_s$ designate the values of the generalized coordinates and momenta on neighboring paths; it is assumed that the latter cross the direct path at the instants t_0 and t_1; then

$$\delta q_s(t_0) = 0, \qquad \delta q_s(t_1) = 0 \qquad (s = 1, \ldots, n) \tag{2.1}$$

The expression of the kinetic potential L on the neighboring path is written down in the form

$$L = \sum_{s=1}^{n} (p_s + \delta p_s)(\dot{q}_s + \delta \dot{q}_s) - H(q_s + \delta q_s,\ p_s + \delta p_s) = L_0 + \delta L + \delta^2 L + \ldots \tag{2.2}$$

here L_0 is the value of L on the direct path. The second variation $\delta^2 L$, if (1.1) is conserved, will be

$$\delta^2 L = \sum_{s=1}^{n} \delta \dot{q}_s \delta p_s - \Omega(\delta q_1, \ldots, \delta q_n, \delta p_1, \ldots, \delta p_n) =$$

$$= \sum_{s=1}^{n} \left[\delta \dot{q}_s \delta p_s - \frac{1}{2}\left(\frac{\partial \Omega}{\partial \delta q_s}\delta q_s + \frac{\partial \Omega}{\partial \delta p_s}\delta p_s\right)\right] = \frac{1}{2}\sum_{s=1}^{n}\left[\delta p_s\left(\delta \dot{q}_s - \frac{\partial \Omega}{\partial \delta p_s}\right) - \right.$$

$$\left. - \delta q_s\left(\delta \dot{p}_s + \frac{\partial \Omega}{\partial \delta q_s}\right)\right] + \frac{1}{2}\frac{d}{dt}\sum_{s=1}^{n} \delta q_s \delta p_s \tag{2.3}$$

It is known that the first variation δS of the action according to Hamilton

$$S = \int_{t_0}^{t_1} L\,dt \tag{2.4}$$

for the neighboring path satisfying Equation (2.1), is equal to zero. Therefore, the second variation $\delta^2 S$, based on (2.3) and (2.1), may be written down in the form

$$\delta^2 S = \frac{1}{2}\int_{t_0}^{t_1}\sum_{s=1}^{n}\left[\delta p_s\left(\delta \dot{q}_s - \frac{\partial \Omega}{\partial \delta p_s}\right) - \delta q_s\left(\delta \dot{p}_s + \frac{\partial \Omega}{\partial \delta q_s}\right)\right] dt \tag{2.5}$$

The second variation will vanish if δq_s and δp_s are solutions of the system of differential equations

$$\delta \dot{q}_s = \frac{\partial \Omega}{\partial \delta p_s}, \qquad \delta \dot{p}_s = -\frac{\partial \Omega}{\partial \delta q_s} \qquad (s = 1, \ldots, n) \tag{2.6}$$

But these are the same differential equations (1.6), which determine the motions along direct paths, infinitely close to the direct path C_0. Therefore, from the first group of conditions (2.1) and Equations (1.11), we have

$$\delta q_s = \sum_{m=1}^{n} u_m(t_0)\,\eta_s^{(m)}(t), \qquad \delta p_s = \sum_{m=1}^{n} u_m(t_0)\,\vartheta_s^{(m)}(t) \tag{2.7}$$

The second group of conditions (2.1) leads to a system of linear homogeneous equations for the determination of the constants $u_m(t_0)$:

$$\sum_{m=1}^{n} u_m(t_0)\,\eta_s^{(m)}(t_1) = 0 \qquad (s = 1, \ldots, n) \tag{2.8}$$

Thus, the determinant (2.9) has to be introduced into the discussion.

$$\Delta(t) = \begin{vmatrix} \eta_1^{(1)}(t) & \ldots & \eta_1^{(n)}(t) \\ \ldots & \ldots & \ldots \\ \eta_n^{(1)}(t) & \ldots & \eta_n^{(n)}(t) \end{vmatrix} \tag{2.9}$$

From (1.10) it follows that $\Delta(t_0) = 0$. Let such a value of t_1^* be found, as t increases, that $\Delta(t)$ is again equal to zero:

$$\Delta(t_1^*) = 0 \tag{2.10}$$

Then the system of equations (2.8) will have a nontrivial solution. There exists a bundle of paths, originating from the initial position $q_s(t_0)$ on the direct path C_0 and intersecting the latter at the position $q_s(t_1^*)$. Along all these paths, which should be considered as being direct, the actions in accordance to Hamilton, calculated within terms of the second order inclusive, are equal to each other (since $\delta^2 S = 0$). The positions $q_s(t_0)$ and $q_s(t_1^*)$ represent the corresponding kinetic foci of simultaneous paths [1]. It is assumed that t_1^* is the first value of $t > t_0$, which makes the determinant (2.9) equal to zero such that

$$\Delta(t) \neq 0 \qquad (t_0 < t < t_1^*) \tag{2.11}$$

This condition, together with the condition regarding the positive definiteness of the quadratic form (1.3)

$$\delta^2 T' = \delta^2 T > 0 \tag{2.12}$$

guarantees the definiteness of the second variation $\delta^2 S$ for an arbitrary neighboring path, originating from the initial configuration $q_2(t_0)$. Therefore, under conditions (2.11) and (2.12), Hamilton's action will be a minimum along the direct path C_0. To prove this statement $\delta^2 L$ is expressed through variations of the generalized coordinates and generalized velocities and the following integral is considered

$$\delta^2 S = \int_{t_0}^{t_1} \delta^2 L\,dt = \int_{t_0}^{t_1} \left[\delta^2 L + \frac{d}{dt}\sum_{s=1}^{n}\sum_{k=1}^{n} \lambda_{sk}\,\delta q_s\,\delta q_k\right] dt \tag{2.13}$$

where $t_1 < t_1^*$. Then it appears to be possible to determine the continuous functions $\lambda_{sk}(t)$, under the indicated conditions, in such a way that the quadratic form under the integral sign (2.13) is positive definite. For $n = 2$ this proof is presented in [5] and [6], while the general case is considered in [7].

Along the direct path C_0, when condition (2.11) is not satisfied, that is when the final configuration $q_s(t_1)$ is attained after passing of the kinetic focus $q_s(t_1{}^*)$, Hamilton's action will not be a minimum since it appears to be possible to construct neighboring paths along which the action will be smaller [8].

3. The most simple example is the one concerning small vibrations of a conservative system about a position of equilibrium. The expressions for the kinetic and potential energy in principal coordinates are

$$T = \frac{1}{2}\sum_{k=1}^{n} \dot{q}_k{}^2 = \frac{1}{2}\sum_{k=1}^{n} p_k{}^2, \qquad \Pi = \frac{1}{2}\sum_{k=1}^{n} \omega_k{}^2 q_k{}^2$$

where ω_k is the frequency of principal oscillations; we have

$$q_s = \alpha_s \cos\omega_s t + \frac{\beta_s}{\omega_s}\sin\omega_s t, \qquad p_s = -\omega_s\alpha_s\sin\omega_s t + \beta_s\cos\omega_s t \quad (3.1)$$

Using (2.9) we obtain

$$\Delta(t) = \frac{1}{\omega_1\omega_2\ldots\omega_n}\sin\omega_1 t\sin\omega_2 t\ldots\sin\omega_n t \quad (3.2)$$

and the closest kinetic focus is reached at the instant of time

$$t_1{}^* = \pi/\omega_n \quad (3.3)$$

where ω_n is the largest frequency. The location of this focus is determined by the generalized coordinates

$$q_s(t_1{}^*) = \alpha_s\cos\frac{\pi\omega_s}{\omega_n} + \frac{\beta_s}{\omega_s}\sin\frac{\pi\omega_s}{\omega_n} \quad (s = 1, \ldots, n-1), \qquad q_n(t_1{}^*) = -\alpha_n \quad (3.4)$$

Hamilton's action (2.4) will be a minimum along the direct path (3.1) only for $0 < t_1 < t_1{}^*$. The interval of time $t_1{}^*$ turns out to be equal to the half-period of principal oscillation with the largest frequency. Therefore, for continuous elastic systems, the variational principle of Hamilton conserves its meaning as an assertion that Hamilton's action is stationary, but not that it is a minimum.

4. Let us consider first the usual scheme regarding the justification of the approximate methods of determining the frequencies and mode shapes, and free vibrations of elastic systems [2], with the aid of the principle of Hamilton-Ostrogradskii.

We limit ourselves to calculation of a system having a finite number of degrees of freedom. In this case the kinetic energy and the potential energy are represented by quadratic forms with constant coefficients

$$T = \frac{1}{2}\sum_{s=1}^{n}\sum_{k=1}^{n} a_{sk}\,\dot{q}_s\dot{q}_k, \qquad \Pi = \frac{1}{2}\sum_{s=1}^{n}\sum_{k=1}^{n} c_{sk}\,q_s q_k \tag{4.1}$$

As direct path we prescribe the motion

$$q_s = C_s \sin \omega t \qquad (s = 1, \ldots, n) \tag{4.2}$$

and we shall assume that the coefficients C_s are varied while the quantity ω is assumed to be given.

Since the variations must vanish at the initial and final instant of time t_0 and t_1, we have to select two instants when $\delta q_s = \delta C_s \sin \omega t$ vanishes, that is, for example, we have to put $t_0 = 0$ and $t_1 = 2\pi/\omega$.

Then on the basis of the principle of Hamilton-Ostrogradskii

$$\delta S = \delta \int_0^{2\pi/\omega} (T - \Pi)\, dt = 0 \tag{4.3}$$

In view of the obvious relations

$$\int_0^{2\pi/\omega} \sin^2 \omega t\, dt = \int_0^{2\pi/\omega} \cos^2 \omega t\, dt = \frac{\pi}{\omega}$$

we find

$$\delta S = \frac{\pi}{\omega}\,\delta(\omega^2 \Gamma - U) = 0 \tag{4.4}$$

where Γ and U are quadratic forms obtained from T and Π respectively, by changing their arguments $\dot{q}_s$ and q_s to C_s.

Sometimes [3,4] it is asserted that on the basis of the principle of Hamilton-Ostrogradskii the quantity $\omega^2\Gamma - U$ must have a minimum value.

In this assertion the following obscurities and contradictions are apparent: 1) the quantity ω is assumed to be given and is not varied, and subsequently deductions are made regarding extremum properties of ω; 2) the integration with respect to time is carried out without accounting for the passage along the direct path of a series of kinetic foci, which makes it impossible to conclude the existence of a minimum.

For systems with distributed constants which possess arbitrarily large natural frequencies an analogous discussion as indicated above is even less applicable.

Thus, from the principle of Hamilton-Ostrogradskii, it is impossible to obtain a justification of relation (4.4) as a variational principle for frequencies and mode shapes.

Let us show that the integral principle expressed by the relation

$$\int_{t_0}^{t_1} \delta L dt = 0 \tag{4.5}$$

which results directly from the general equation of dynamics permits the satisfaction of expression (4.4) as a variational principle for eigenvalues. For fixed values of t_0 and t_1 the form (4.5) is obviously equivalent to the form (4.3). If, however, t_0 and t_1 depend on the varied quantities then

$$\delta S = \int_{t_0}^{t_1} \delta L dt + (L)_{t=t_1} \delta t_1 - (L)_{t=t_0} \delta t_0 \tag{4.6}$$

where δt_0 and δt_1 are variations of the limits of integration. Then (4.5) takes on the form [8,9]

$$\delta S + (L)_{t=t_0} \delta t_0^* - (L)_{t=t_1} \delta t_1 = 0 \tag{4.7}$$

Let us now prescribe a neighboring path in the form

$$q_s'(t) = C_s' \sin(\omega' t + \alpha') \tag{4.8}$$

such that we shall assume that the quantities C_s', ω' and a' are infinitely close to quantities $C_s \omega$ and a, respectively, which correspond to the direct path

$$q_s(t) = C_s \sin(\omega t + \alpha) \tag{4.9}$$

Then

$$q_s'(t) = q_s(t) + \delta C_s \sin(\omega t + \alpha) + \delta\alpha C_s \cos(\omega t + \alpha) + C_s t\, \delta\omega \cos(\omega t + \alpha) + \ldots \tag{4.10}$$

Let us now assume that the limits of integration differ by one period, that is, $t_1 = t_0 + 2\pi/\omega$ and

$$\delta t_1 = \delta t_0 - \frac{2\pi}{\omega^2} \delta\omega \tag{4.11}$$

Substituting into (4.5) the value

$$\delta L = \sum_{s=1}^{n} \left(\frac{\partial L}{\partial q_s} \delta q_s + \frac{\partial L}{\partial \dot{q}_s} \delta \dot{q}_s \right)$$

and carrying out the integration by parts we have

$$\sum_{s=1}^{n} \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial q_s} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_s} \right] \delta q_s dt + \sum_{s=1}^{n} \left[\left(\frac{\partial L}{\partial \dot{q}_s} \right)_{t=t_1} \delta q_s(t_1) - \left(\frac{\partial L}{\partial \dot{q}_s} \right)_{t=t_0} \delta q_s(t_0) \right] = 0 \tag{4.12}$$

If, as it is in our case, $\partial L/\partial q_s$ is a periodic function, then to obtain from (4.12) the equations of motion it is sufficient to satisfy the conditions

$$\delta q_s(t_1) - \delta q_s(t_0) = 0 \qquad (s = 1, \ldots, n) \tag{4.13}$$

without letting the variations at the ends be equal to zero separately.

In view of the periodicity of L the integral principle (4.7) may be represented in the form

$$\delta S + (L)_{t=t_0} \frac{2\pi}{\omega^2} \delta\omega = 0 \tag{4.14}$$

From (4.10) we have

$$\delta q_s(t_1) = \delta C_s \sin(\omega t_0 + \alpha) + C_s \delta\alpha \cos(\omega t_0 + \alpha) + C_s t_1 \delta\omega \cos(\omega t_0 + \alpha)$$
$$\delta q_s(t_0) = \delta C_s \sin(\omega t_0 + \alpha) + C_s \delta\alpha \cos(\omega t_0 + \alpha) + C_s t_0 \delta\omega \cos(\omega t_0 + \alpha)$$

and since $t_1 - t_0 \neq 0$, we have to assume in order to satisfy the conditions (4.13) on the neighboring paths

$$\omega t_0 + \alpha = \frac{1}{2}\pi \tag{4.15}$$

The expression L, after substitution of the values of the generalized coordinates (4.9), takes on the form

$$L = \omega^2 \Gamma \cos^2(\omega t + \alpha) - U \sin^2(\omega t + \alpha) \tag{4.16}$$

where Γ and U have the same values as in Equation (4.4). Then

$$S = \int_{t_0}^{t_1} L dt = \frac{\pi}{\omega}(\omega^2 \Gamma - U) \tag{4.17}$$

and further

$$\delta S = \pi\left(\Gamma + \frac{U}{\omega^2}\right)\delta\omega + \frac{\pi}{\omega}(\omega^2 \delta\Gamma - \delta U) \tag{4.18}$$

Noting further that by (4.14) and (4.15)

$$(L)_{t=t_0} \frac{2\pi}{\omega^2} \delta\omega = -U \frac{2\pi}{\omega^2} \delta\omega \tag{4.19}$$

we may write down relation (4.18) in the form

$$\left(\Gamma - \frac{U}{\omega^2}\right)\delta\omega + \frac{1}{\omega}(\omega^2 \delta\Gamma - \delta U) = 0 \tag{4.20}$$

For a direct path the maximum value of kinetic energy $\omega^2\Gamma$ is equal to the maximum value of potential energy U on the basis of the law of conservation of energy:

$$\omega^2 \Gamma = U \tag{4.21}$$

such that relationship (4.20) may be represented in the form

$$\omega^2 \delta\Gamma - \delta U = 0 \tag{4.22}$$

Therefore, if we introduce into our consideration the quantity

$$R = \frac{1}{\pi}\,\omega S = \omega^2\Gamma - U \qquad (4.23)$$

then its variation $\delta' R$, calculated with a fixed ω, will be equal to zero:

$$\delta' R = \omega^2 \delta\Gamma - \delta U = 0 \qquad (4.24)$$

We note that (4.24) is a condition for a stationary value of the functional U in the presence of a supplementary requirement $\Gamma = 1$; then ω^2 plays the role of a Lagrange multiplier. Such an approach to the functional (4.23), characteristic of problems for determination of eigenvalues, explains the vanishing of the term with $\delta\omega$ in the preceding calculations.

Condition (4.24) may be used to obtain equations which permit the approximate determination of the frequencies and mode shapes; however, to judge the extremum properties of frequencies and mode shapes, supplementary considerations have to be invoked which are not connected with variational and integral principles of mechanics.

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